

# A Lattice Gauge Model of Singular Marsden-Weinstein Reduction

## Part I. Kinematics

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### Abstract

The simplest nontrivial toy model of a classical  $SU(3)$  lattice gauge theory is studied in the Hamiltonian approach. By means of singular symplectic reduction, the reduced phase space is constructed. Two equivalent descriptions of this space in terms of a symplectic covering as well as in terms of invariants are derived.

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## 1 Introduction

In the study of quantum gauge theory by nonperturbative methods there exist, in effect, two approaches: one is to quantize the unreduced system and then reduce the symmetries on the quantum level, the other one is to first reduce the symmetries on the classical

level and then quantize the reduced system. For a discussion of the first strategy within the framework of lattice gauge theory, see [10] and the references therein. The aim of the present paper is to contribute to the second approach. The motivation behind stems from the well-known fact that nonabelian gauge fields can have several symmetry types, which give rise to singularities in the 'true' (i.e., reduced) classical configuration space. Mathematically spoken, the latter is a stratified space rather than a smooth manifold. It is natural to ask whether the singularities produce physical effects. For a systematic study of this open problem one needs a concept of how to implement the singularity structure in quantum theory. Such concepts have been developed in recent years, see, e.g., [6, 7, 11]. To separate the problem of symmetry reduction from the usual problems of a field theory related to the infinite number of degrees of freedom, it is reasonable to first study lattice gauge theory. This way, one obtains a variety of toy models to form and test concepts and to develop quantum theory on a space with singularities. It is important for quantum theory, as well as interesting in its own right, to understand the classical dynamics of these models. Thus, in the present paper we consider the simplest nontrivial model of an  $SU(3)$  lattice gauge theory, where the lattice consists of a single plaquette. We study the kinematics of this model, i.e., the structure of the reduced phase space. The classical dynamics will then be studied in a subsequent paper.

We proceed as follows. In Section 2 we introduce the model. In Section 3 we carry out symmetry reduction. This will lead us to the so-called reduced cotangent bundle [12]. Then we give two equivalent descriptions of this bundle. One is in terms of a symplectic covering (Section 4), the other one is in terms of invariants (Section 5). We conclude with some general remarks on the dynamics in Section 6.

## 2 The model

Let us consider chromodynamics on a finite regular cubic lattice  $\Lambda$ . Denote the set of the oriented,  $i$ -dimensional elements of  $\Lambda$  by  $\Lambda^i$  (sites, links, plaquettes and cubes in increasing order of  $i$ ). The gauge group is  $G = SU(3)$ , its Lie algebra is  $\mathfrak{g} = \mathfrak{su}(3)$ . The classical gluonic potential is approximated by its parallel transporter:

$$\Lambda^1 \ni (x, y) \mapsto a_{(x,y)} \in G.$$

Thus, the unreduced classical configuration space is the direct product  $G^{\Lambda^1}$  and the unreduced phase space is  $T^*G^{\Lambda^1}$ . By means of the natural isomorphisms  $T^*G^{\Lambda^1} \cong (T^*G)^{\Lambda^1}$ ,  $T^*G \cong G \times \mathfrak{g}^*$  and  $\mathfrak{g}^* \cong \mathfrak{g}$ , see below for details, the canonically conjugate momenta (colour electric fields) are given by maps

$$\Lambda^1 \ni (x, y) \mapsto A_{(x,y)} \in \mathfrak{g}.$$

Local gauge transformations are approximated by maps

$$\Lambda^0 \ni x \mapsto g_x \in G,$$

hence the group of local gauge transformations is the direct product  $G^{\Lambda^0}$ . It acts on the phase space as follows:

$$a'_{(x,y)} = g_x \cdot a_{(x,y)} \cdot g_y^{-1}, \quad A'_{(x,y)} = g_x \cdot A_{(x,y)} \cdot g_x^{-1}.$$

The (gauge invariant) Hamiltonian is given by

$$H = -\frac{\delta^3}{2} \sum_{(x,y) \in \Lambda^1} \text{tr}(A_{(x,y)}^2) + \frac{1}{2g^2\delta} \sum_{p \in \Lambda^2} (6 - \text{tr}(a_p + a_p^\dagger)). \quad (1)$$

Here,  $\delta$  and  $g$  denote the lattice spacing and the coupling constant, respectively, and  $a_p$  is the parallel transporter around the plaquette  $p$ . For a plaquette  $p$  with vertices  $x, y, z, u$  we choose

$$a_p = a_{xy} \cdot a_{yz} \cdot a_{zu} \cdot a_{ux}.$$

While  $a_p$  depends on the choice of a base point  $x$ ,  $\text{tr}(a_p)$  does not.

In the present paper we consider the case where  $\Lambda$  consists of a single plaquette. This is the simplest nontrivial model for a Hamiltonian lattice gauge theory. On three of the links of the plaquette,  $a$  and  $A$  can be gauged to  $\mathbb{1}$  and  $0$ , respectively. Such a gauge is called a tree gauge. Then the residual gauge freedom consists of constant gauge transformations. Thus, the unreduced configuration space is the group manifold  $G$  and the unreduced phase space is  $T^*G \cong G \times \mathfrak{g}$ . Its elements will be denoted by  $(a, A)$ . The gauge group is  $G$ , its action on the phase space is given by diagonal conjugation

$$a' = gag^{-1}, \quad A' = gAg^{-1}.$$

The Hamiltonian becomes

$$H = -\frac{\delta^3}{2} \text{tr}(A^2) + \frac{1}{2g^2\delta} (6 - \text{tr}(a + a^\dagger)). \quad (2)$$

Next, we will carry out symmetry reduction. The basic object for this is the  $G$ -manifold of the unreduced configuration space, because it determines the kinematical structure of the model completely.

### 3 Symmetry reduction

First, let us recall the general procedure. It is known as cotangent bundle reduction and is a special case of (singular) Marsden-Weinstein reduction.

#### 3.1 Cotangent bundle reduction

Let  $Q$  be a manifold acted upon properly by a Lie group  $K$  (we may even assume that  $K$  is compact). Associated with  $(Q, K)$  there is the surjection

$$\pi : T^*(Q/K) \rightarrow Q/K. \quad (3)$$

The base space  $Q/K$  consists of the  $K$ -orbits in  $Q$ , equipped with the quotient topology, the stratification by the orbit types of  $K$ -action and the smooth structure

$$C^\infty(Q/K) := C^\infty(Q)^K$$

(invariant smooth functions on  $Q$ ). Thus,  $Q/K$  is a stratified topological space with smooth structure, see [15] for this notion.

The total space  $T^*(Q/K)$  is obtained as follows. The action of  $K$  on  $Q$  lifts to a proper symplectic action of  $K$  on the cotangent bundle  $T^*Q$  by the corresponding point transformations. The map  $J : T^*Q \rightarrow \mathfrak{k}^*$  defined by

$$\langle J(\eta), X \rangle := \eta(X^Q), \quad \eta \in T^*Q, \quad X \in \mathfrak{k}, \quad (4)$$

where  $X^Q$  denotes the Killing vector field associated with  $X$ , is an equivariant momentum mapping for this action [1, §4.2]. (Thus, these data define a Hamiltonian  $G$ -manifold naturally associated with  $(Q, K)$ .) Since  $J$  is equivariant, the level set  $J^{-1}(0)$  is invariant under  $K$ . The bundle space  $T^*(Q/K)$  is given by the topological quotient  $J^{-1}(0)/K$ . It is equipped with the following structure, see [2, 13, 18] or [5, App. B.5]:

– A smooth Poisson structure. The natural smooth structure of  $T^*(Q/K)$  is given by

$$C^\infty(T^*(Q/K)) := C^\infty(T^*Q)^K / V^K,$$

where  $V$  denotes the vanishing ideal of the level set  $J^{-1}(0)$ . Since  $K$  acts symplectically on  $T^*Q$ ,  $C^\infty(T^*Q)^K$  is a Poisson subalgebra of  $C^\infty(T^*Q)$ . In view of Noether's theorem,  $J^{-1}(0)$  is invariant under the Hamiltonian flow of invariant functions. Hence,  $V^K$  is a Poisson ideal in  $C^\infty(T^*Q)^K$ . Therefore,  $C^\infty(T^*(Q/K))$  inherits a Poisson bracket through

$$\{f + V^K, g + V^K\}_{T^*(Q/K)} = \{f, g\}_{T^*Q}, \quad f, g \in C^\infty(T^*(Q/K)).$$

– A stratification by orbit types. Using the slice theorem it can be shown that for given orbit type  $\tau$  the subset  $J^{-1}(0)_\tau$  of  $J^{-1}(0)$  consisting of the elements of type  $\tau$  is an embedded submanifold of  $T^*Q$ . Local charts on the  $\tau$ -stratum  $T^*(Q/K)_\tau$  of  $T^*(Q/K)$  are then defined in the usual way: for a given point in  $T^*(Q/K)_\tau$  one chooses a representative in  $J^{-1}(0)_\tau$  and a slice about the representative for the action of  $K$  on  $J^{-1}(0)_\tau$ . By restriction, the natural projection  $\pi_\tau : J^{-1}(0)_\tau \rightarrow T^*(Q/K)_\tau$  induces a homeomorphism of the slice onto its image. Thus, charts on the slice induce charts on  $T^*(Q/K)_\tau$ .

– Symplectic structures on the strata  $T^*(Q/K)_\tau$ . One can prove that the annihilator of the pull-back of the symplectic form  $\omega$  of  $T^*Q$  to the submanifold  $J^{-1}(0)_\tau$  coincides with the distribution defined by the tangent spaces of the orbits. Therefore, the pull-back of  $\omega$  to a slice for the action of  $K$  on  $J^{-1}(0)_\tau$  is a symplectic form on that slice. Through the homeomorphism of the slice onto its image in  $T^*(Q/K)_\tau$ , induced by the natural projection

$\pi_\tau$  it defines a local symplectic form on  $T^*(Q/K)_\tau$ . Due to the fact that  $\omega$  is  $K$ -invariant, all the local forms merge to a symplectic form  $\omega^\tau$  on  $T^*(Q/K)_\tau$ . Then

$$\pi_\tau^* \omega^\tau = j_\tau^* \omega,$$

where  $j_\tau : J^{-1}(0)_\tau \rightarrow T^*Q$  denotes the natural injection.

By construction, the injections  $(T^*Q)_\tau \rightarrow T^*(Q/K)$  are Poisson maps. Therefore, the above data turn  $T^*(Q/K)$  into a stratified symplectic space.

Finally, the projection  $\pi$  of (3) is induced by the restriction of the natural (equivariant) projection  $T^*Q \rightarrow Q$  to the level set  $J^{-1}(0)$ . Since  $J^{-1}(0)$  contains the zero section of  $T^*Q$ ,  $\pi$  is surjective.

**Remark 3.1.** The fibres of (3) may intersect several distinct strata of  $T^*(Q/K)$ . In particular,  $\pi$  does not preserve the orbit types. However, as the stabilizer of a covector in  $T^*Q$  cannot be larger than that of its base point,  $\pi$  does not decrease orbit types. For a detailed study of the stratifications of the fibres of  $T^*(Q/K)$ , see [14].

**Remark 3.2.** Since (3) is a bundle in the topological category in the sense of [9] and since it plays the same role for  $Q/K$  as the cotangent bundle  $T^*Q$  plays for  $Q$ , (3) is called the reduced cotangent bundle in [12], although in general its elements are not covectors. When  $K$  acts freely then  $Q/K$  is a manifold and (3) is isomorphic to the cotangent bundle of this manifold [1]. In general, the cotangent bundles of the strata of  $Q/K$  are dense subsets of the corresponding strata of  $T^*(Q/K)$  [14].

If, like in our case,  $(Q, K)$  is the configuration space of a Hamiltonian system with symmetries,  $Q/K$  and  $T^*(Q/K)$  are referred to as the reduced configuration space and the reduced phase space, respectively. It can be shown in general [13] that if an evolution curve in  $T^*Q$  w.r.t. a  $K$ -invariant Hamiltonian meets a submanifold  $J^{-1}(0)_\tau$  then it is contained completely in this submanifold. Therefore, dynamics in  $T^*(Q/K)$  takes place inside the strata. Due to Remark 3.1, an analogous statement for  $Q/K$  is in general not true, though.

We will now discuss the reduced data of our model in detail. The reduced configuration space  $Q/K$  and the reduced phase space  $T^*(Q/K)$  will be denoted by  $\mathcal{X}$  and  $\mathcal{P}$ , respectively.

## 3.2 The reduced configuration space $\mathcal{X}$

In what follows we will write  $G$  for  $SU(3)$  and  $\mathfrak{g}$  for  $\mathfrak{su}(3)$ .

By construction,  $\mathcal{X}$  is the adjoint quotient  $G/\text{Ad}$ . As  $G$  is semisimple, this space has the following two standard realizations. Let  $T$  denote the subgroup of diagonal matrices of  $G$ . One has  $T \cong U(1) \times U(1)$ , a 2-torus. For  $j = 1, 2, 3$ , let  $T_{(j)}$  denote the subsets of  $T$  consisting of the elements whose entries coincide, possibly except for the  $j$ th one. Let  $\mathcal{A}$  denote one of the triangular subsets of  $T$  which are cut out by the  $T_{(j)}$ ,  $j = 1, 2, 3$ , see Figure 1. From the embedding  $\mathcal{A} \rightarrow T$ ,  $\mathcal{A}$  acquires a Whitney smooth structure. It

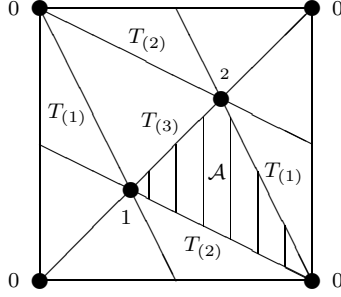


Figure 1: A possible choice for the subset  $\mathcal{A}$  of  $T$ . The numbers 0, 1, 2 stand for the central elements  $\mathbb{1}$ ,  $e^{i\frac{2}{3}\pi}\mathbb{1}$  and  $e^{i\frac{4}{3}\pi}\mathbb{1}$ , respectively.

is a standard fact that the embeddings  $\mathcal{A} \rightarrow T \rightarrow G$  induce, by passing to quotients, isomorphisms

$$\mathcal{X} \cong T/S_3 \cong \mathcal{A} \quad (5)$$

of topological spaces with smooth structure. Here the symmetric group  $S_3$  acts by permutation of entries and the smooth structure of  $T/S_3$  is defined by the invariant smooth functions on  $T$ .

Let us describe the stratification. The number of distinct entries of  $a \in \mathcal{A}$  can be 3, 2 or 1. Denote the corresponding subsets of  $\mathcal{A}$  by  $\mathcal{A}_k$  with  $k = 2, 1, 0$ . One has  $\mathcal{A}_1 = \bigcup_{j=1}^3 \mathcal{A} \cap T_{(j)}$ . Topologically,  $\mathcal{A}$  is a 2-simplex,  $\mathcal{A}_2$  is its interior,  $\mathcal{A}_1$  consists of the edges without the vertices and  $\mathcal{A}_0$  consists of the vertices. Taking into account that the stabilizer of  $a$  under the action of  $\text{SU}(3)$  is given by the centralizer of  $a$  in  $\text{SU}(3)$ , the stabilizer of  $a \in \mathcal{A}_k$  is

$k$	$\text{SU}(3)$ -stabilizer	$S_3$ -stabilizer
2	$T$	$\{\mathbb{1}\}$
1	$\text{U}(2)$	$S_2$
0	$\text{SU}(3)$	$S_3$

(6)

In particular,  $\mathcal{A}_0 = \mathbb{Z}_3$ , the centre of  $\text{SU}(3)$ . Denote the orbit types in the respective order by  $\tau_2$ ,  $\tau_1$  and  $\tau_0$ , irrespective of the action they belong to, and the corresponding strata of  $\mathcal{X}$  by  $\mathcal{X}_2$ ,  $\mathcal{X}_1$  and  $\mathcal{X}_0$ . (The numbering refers to the dimensions of the strata.) Type  $\tau_2$  is the principal orbit type and  $\mathcal{X}_2$  is the principal stratum.

It is easy to see that the isomorphism  $\mathcal{X} \cong \mathcal{A}$  holds on the level of stratified smooth topological spaces.

**Remark 3.3.** The identification of  $\mathcal{X}$  with  $\mathcal{A}$  endows  $\mathcal{X}$  with a CW-complex structure in an obvious fashion. Already for the quotient  $(\text{SU}(3) \times \text{SU}(3))/\text{SU}(3)$  with  $\text{SU}(3)$  acting by diagonal conjugation, which is the reduced configuration space of lattice  $\text{SU}(3)$ -gauge theory on a lattice with 2 plaquettes, the construction of a CW-complex structure is much more complicated, see [4].

### 3.3 The reduced phase space $\mathcal{P}$

As anticipated in Section 2, we identify  $T^*G$  with the direct product  $G \times \mathfrak{g}$  by virtue of the natural diffeomorphism

$$G \times \mathfrak{g} \rightarrow T^*G, \quad (a, A) \mapsto \langle A, R'_{a^{-1}} \cdot \rangle. \quad (7)$$

Here,  $R_a : G \rightarrow G$  denotes right multiplication by  $a \in G$  and  $\langle \cdot, \cdot \rangle$  is the ordinary scalar product of complex matrices,

$$\langle A, B \rangle = \text{tr}(A^\dagger B), \quad A, B \in M_3(\mathbb{C}).$$

When restricted to  $\mathfrak{g}$  this form yields a real scalar product which, up to a constant factor, coincides with the negative of the Killing form of  $\mathfrak{g}$ :

$$\langle A, B \rangle = -\text{tr}(AB), \quad A, B \in \mathfrak{g}.$$

Since  $T(G \times \mathfrak{g}) \cong TG \times T\mathfrak{g}$ , vectors tangent to  $G \times \mathfrak{g}$  at  $(a, A)$  can be written as  $(R'_a B, (A, C))$  with  $B, C \in \mathfrak{g}$ . Under the identification (7) the symplectic potential of  $T^*G$  takes the standard form

$$\theta_{(a,A)}(R'_a B, (A, C)) = \langle A, B \rangle, \quad (8)$$

hence the symplectic form  $\omega = d\theta$  is

$$\omega_{(a,A)}\left((R'_a B_1, (A, C_1)), (R'_a B_2, (A, C_2))\right) = \langle B_1, C_2 \rangle - \langle C_1, B_2 \rangle - \langle A, [B_1, B_2] \rangle. \quad (9)$$

The action of  $G$  on  $T^*G$  by the induced point transformations is given by conjugation, i.e.,

$$b \cdot (a, A) = (b a b^{-1}, b A b^{-1}). \quad (10)$$

If we furthermore identify  $\mathfrak{g}^*$  with  $\mathfrak{g}$  by virtue of the scalar product  $\langle \cdot, \cdot \rangle$ , the natural momentum mapping for this action is given by the map

$$J : G \times \mathfrak{g} \rightarrow \mathfrak{g}, \quad J(a, A) = A - a^{-1} A a. \quad (11)$$

The level set  $J^{-1}(0)$  is therefore given by all pairs  $(a, A) \in G \times \mathfrak{g}$  where  $a$  and  $A$  commute. In particular, it contains the subset  $T \times \mathfrak{t}$ . By restriction of the natural projection to orbits we obtain a map

$$\lambda : T \times \mathfrak{t} \rightarrow \mathcal{P}. \quad (12)$$

Let  $(a, A) \in J^{-1}(0)$ . Since  $a$  and  $A$  commute, they possess a common eigenbasis. Since  $a$  is unitary and  $A$  is anti-Hermitian, the eigenbasis can be chosen to be orthonormal. Hence, by  $G$ -action,  $(a, A)$  can be transported to  $T \times \mathfrak{t}$ . In other words, every  $G$ -orbit in  $J^{-1}(0)$  intersects the subset  $T \times \mathfrak{t}$ . Hence,  $\lambda$  is surjective. Since two elements of  $T \times \mathfrak{t}$  are conjugate under  $G$  iff they differ by a simultaneous permutation of their entries, then  $\lambda$  descends to a bijection

$$(T \times \mathfrak{t})/S_3 \rightarrow \mathcal{P}.$$

Standard arguments ensure that this is in fact a homeomorphism. Thus, we can use  $\lambda$  to describe  $\mathcal{P}$ . In particular,  $\mathcal{P}$  is an orbifold.

We start with the stratification. The number of entries which simultaneously coincide for both  $a$  and  $A$  can be 0, 2 or 3. Denote the corresponding subsets of  $T \times \mathfrak{t}$  by  $(T \times \mathfrak{t})_k$  with  $k = 2, 1, 0$ , respectively. The stabilizers and orbit types of  $(a, A) \in (T \times \mathfrak{t})_k$  under  $SU(3)$ -action and  $S_3$ -action are

$k$	$SU(3)$ -stabilizer	$S_3$ -stabilizer	orbit type
2	$T$	$\{\mathbb{1}\}$	$\tau_2$
1	$U(2)$	$S_2$	$\tau_1$
0	$SU(3)$	$S_3$	$\tau_0$

(13)

Since the orbit types are the same as for  $\mathcal{X}$  we use the same notation. Let  $\mathcal{P}_k \subseteq \mathcal{P}$  denote the stratum of type  $\tau_k$ ,  $k = 0, 1, 2$ .  $\mathcal{P}_2$  is the principal stratum. Since the subsets  $(T \times \mathfrak{t})_k$  are the pre-images of the strata  $\mathcal{P}_k$  under  $\lambda$ , they will be referred to as strata of  $T \times \mathfrak{t}$ . By restriction,  $\lambda$  induces maps

$$\lambda_k : (T \times \mathfrak{t})_k \rightarrow \mathcal{P}_k, \quad k = 2, 1, 0, \quad (14)$$

which descend to homeomorphisms of  $(T \times \mathfrak{t})_k/S_3$  onto  $\mathcal{P}_k$ ,  $k = 2, 1, 0$ .

We determine  $(T \times \mathfrak{t})_k$  explicitly. Recall that  $\mathbb{Z}_3$  denotes the center of  $G = SU(3)$ . As for  $T_{(j)}$ , let  $\mathfrak{t}_{(j)}$ ,  $j = 1, 2, 3$ , denote the subset of  $\mathfrak{t}$  consisting of the elements whose entries coincide, possibly except for the  $j$ th one. We find

$$\begin{aligned} (T \times \mathfrak{t})_0 &= \mathbb{Z}_3 \times \{0\}, \\ (T \times \mathfrak{t})_1 &= \left( \bigcup_{j=1}^3 T_{(j)} \times \mathfrak{t}_{(j)} \right) - (T \times \mathfrak{t})_0, \\ (T \times \mathfrak{t})_2 &= T \times \mathfrak{t} - (T \times \mathfrak{t})_1. \end{aligned}$$

These are embedded submanifolds of  $T \times \mathfrak{t}$ . Since  $\mathfrak{t}$  is the Lie subalgebra of  $\mathfrak{g}$  associated with the Lie subgroup  $T$  of  $G$ ,  $T \times \mathfrak{t}$  is a *symplectic* submanifold of  $G \times \mathfrak{g}$ . Analogously, so are  $T_{(j)} \times \mathfrak{t}_{(j)}$ ,  $j = 1, 2, 3$ . It follows that  $(T \times \mathfrak{t})_k$ ,  $k = 2, 1$ , are symplectic manifolds. For convenience, in the following we will view  $(T \times \mathfrak{t})_0$  as a (trivially) symplectic manifold, too.

**Theorem 3.4.** *The map  $\lambda$  is Poisson. The maps  $\lambda_k$  are local symplectomorphisms.*

*Proof.* By definition,  $C^\infty(\mathcal{P})$  is a quotient of  $C^\infty(G \times \mathfrak{g})^G$ . Hence, the first assertion is a direct consequence of the fact that  $T \times \mathfrak{t}$  is a symplectic submanifold of  $G \times \mathfrak{g}$ . For the second assertion, recall the construction of the symplectic forms on the strata  $\mathcal{P}_k$  from Subsection 3.1. The assertion then follows by observing that any point of  $\mathcal{P}_k$  has a representative in  $(T \times \mathfrak{t})_k$  and that a sufficiently small neighbourhood of the chosen representative in  $(T \times \mathfrak{t})_k$  provides a slice for the action of  $G$  on the submanifold  $J^{-1}(0)_k$  of  $G \times \mathfrak{g}$ . Here  $J^{-1}(0)_k$  denotes the subset of  $J^{-1}(0)$  consisting of the elements of the orbits of type  $\tau_k$ .  $\square$



**Remark 3.5.** 1. Since the submanifolds  $(T \times \mathfrak{t})_k$  are symplectic and since  $S_3$  is finite, the quotient  $(T \times \mathfrak{t})/S_3$  naturally carries the structure of a stratified symplectic space. Of course, this structure might be viewed as to be obtained by singular Marsden-Weinstein reduction with (necessarily) trivial momentum map. Then Theorem 3.4 says that the map  $\lambda$  induces an isomorphism of stratified symplectic spaces of  $(T \times \mathfrak{t})/S_3$  onto  $\mathcal{P}$ .

2. Dynamics on  $\mathcal{P}$  is thus given by the dynamics on  $T \times \mathfrak{t}$  w.r.t. an  $S_3$ -invariant Hamiltonian and the symplectic form (9). Similarly, motion on  $\mathcal{X}$  is given by  $S_3$ -invariant motion on the 2-torus with metric defined by the scalar product  $\langle \cdot, \cdot \rangle$ .

### 3.4 The projection $\pi : \mathcal{P} \rightarrow \mathcal{X}$

Recall from Subsection 3.1 that the projection  $\pi : \mathcal{P} \rightarrow \mathcal{X}$  is induced by the cotangent bundle projection  $T^*G \rightarrow G$ . By virtue of the identification (7), the latter is identified with the natural projection to the first factor  $\text{pr}_1 : G \times \mathfrak{g} \rightarrow G$ . Hence, one has the following commutative diagram

$$\begin{array}{ccc} T \times \mathfrak{t} & \xrightarrow{\lambda} & \mathcal{P} \\ \text{pr}_1 \downarrow & & \downarrow \pi \\ T & \longrightarrow & \mathcal{X} \end{array}$$

where the lower horizontal arrow is defined by restriction of the natural projection  $G \rightarrow \mathcal{X}$ . It follows that the fibre over  $a \in \mathcal{X}$  ( $\mathcal{X}$  being identified with  $\mathcal{A}$  and hence with a subset of  $T$ ) is given by

$$\pi^{-1}(a) = \mathfrak{t}/S(a),$$

where  $S(a)$  is the stabilizer of  $a$  under the action of  $S_3$ . According to (6), there are 3 cases, illustrated in Figure 2.

– If  $a \in \mathcal{X}_2$ ,  $S(a)$  is trivial, hence  $\pi^{-1}(a) = \mathfrak{t}$ . I.e., the fibre is a full 2-plane and belongs to the stratum  $\mathcal{P}_2$ .

– If  $a \in \mathcal{X}_1$  then  $a \in T_{(j)} - \mathbb{Z}_3$  for some  $j = 1, 2, 3$ . Then  $S(a) = S_2$ , acting by permuting the 2 entries besides the  $j$ th one. Hence,  $\pi^{-1}(a) = \mathfrak{t}/S_2$ , acting by reflection about the subspace  $\mathfrak{t}_{(j)}$ . Therefore, the fibre may be identified with one of the two half-planes of  $\mathfrak{t}$  cut out by  $\mathfrak{t}_{(j)}$ . Its interior belongs to the stratum  $\mathcal{P}_2$ , whereas the boundary  $\mathfrak{t}_{(j)}$  belongs to the stratum  $\mathcal{P}_1$ .

– If  $a \in \mathcal{X}_0$ , i.e.,  $a \in \mathbb{Z}_3$ , then  $S(a) = S_3$ . The action of  $S_3$  on  $\mathfrak{t}$  is generated by the reflections about the 3 subspaces  $\mathfrak{t}_{(j)}$ ,  $j = 1, 2, 3$ . Hence, the fibre may be identified with one of the 6 (closed) Weyl chambers of  $\mathfrak{t}$  cut out by  $\mathfrak{t}_{(j)}$ ,  $j = 1, 2, 3$  (the walls of the Weyl chambers). The interior of the Weyl chamber belongs to the stratum  $\mathcal{P}_2$ , the walls minus the origin belong to the stratum  $\mathcal{P}_1$  and the origin belongs to the stratum  $\mathcal{P}_0$ .

One can see explicitly that the projection  $\pi : \mathcal{P} \rightarrow \mathcal{X}$  does not preserve the stratification, because the fibres over points in  $\mathcal{X}_1$  and  $\mathcal{X}_0$  intersect more than 1 stratum of  $\mathcal{P}$ . As stated in Remark 3.1, this is a general phenomenon.

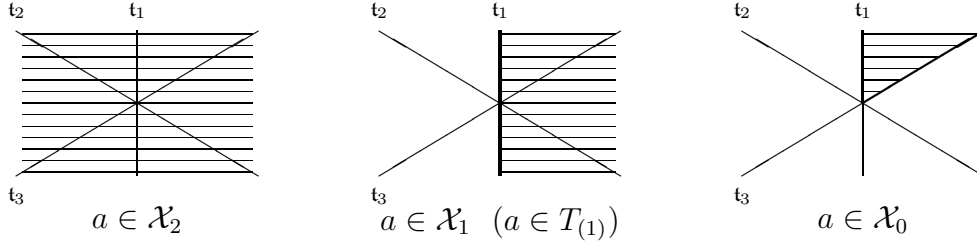


Figure 2: The fibres  $\pi^{-1}(a)$ .

**Remark 3.6.** The description of the reduced data given here generalizes to an arbitrary compact semisimple Lie group in an obvious way:  $T$  and  $\mathfrak{t}$  are replaced by a maximal torus in  $G$  and its Lie algebra, which is a Cartan subalgebra of  $\mathfrak{g}$ .  $\mathcal{A}$  is replaced by a Weyl alcove in  $T$  and  $S_3$  is replaced by the Weyl group of  $G$ . It is interesting that for  $G = \mathrm{SU}(2)$  one obtains the reduced phase space of the spherical pendulum with zero angular momentum, which is the well-known canoe [5, §VI.2].

This completes the construction of the reduced data for the model under consideration. Next, we will derive tools to study the dynamics of this model. I.e., first, a symplectic covering of  $T \times \mathfrak{t}$  and, second, a description of  $\mathcal{P}$  and  $\mathcal{X}$  in terms of invariants.

## 4 Symplectic covering of $T \times \mathfrak{t}$

Recall the symplectic form  $\omega$  of  $G \times \mathfrak{g}$ , see (9). By an abuse of notation, the pull-back of this form to  $T \times \mathfrak{t}$  will also be denoted by  $\omega$ . Elements of  $\mathbb{R}^4$  will be denoted by  $(x, p) \equiv ((x^1, x^2), (p_1, p_2))$ . In this section, we use the exponential map of  $T$  to construct a covering  $\psi : \mathbb{R}^4 \rightarrow T \times \mathfrak{t}$  which pulls back  $\omega$  to the natural symplectic form  $dp_i \wedge dx^i$  of  $\mathbb{R}^4$  (summation convention). We choose  $\psi$  to be induced by some covering  $\varphi : \mathbb{R}^2 \rightarrow T$  by virtue of the commutative diagram

$$\begin{array}{ccc}
 T\mathbb{R}^2 & \xrightarrow{\varphi'} & TT \\
 g \downarrow & & \downarrow h \\
 \mathbb{R}^4 \cong T^*\mathbb{R}^2 & \xrightarrow{\psi} & T^*T \cong T \times \mathfrak{t}
 \end{array} \tag{15}$$

where the vertical arrows stand for the isomorphisms between the tangent and cotangent bundles induced by the natural Riemannian metrics  $g$  on  $\mathbb{R}^2$  and  $h$  on  $T$ . Recall that  $h$  is given by the restriction to  $T$  of the Killing metric of  $G$  induced by the scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ . A straightforward calculation, where  $\mathbb{R}^2$  and  $T$  may be replaced by arbitrary Riemannian manifolds, shows that if  $\varphi$  is isometric then  $\psi$  is symplectic. Thus, all we have to do is to choose  $\varphi$  appropriately. E.g., we can choose  $\varphi$  as the composition of the isomorphism  $\mathbb{R}^2 \rightarrow \mathfrak{t}$ , mapping the canonical basis vectors  $e_1, e_2$  to the orthonormal basis

$$\mathrm{diag}\left(\frac{i}{\sqrt{6}}, \frac{i}{\sqrt{6}}, -i\sqrt{\frac{2}{3}}\right), \quad \mathrm{diag}\left(\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0\right)$$

in  $\mathfrak{t}$ , with the exponential map  $\mathfrak{t} \rightarrow T$ :

$$\varphi(x) = \text{diag} \left( e^{i(\frac{1}{\sqrt{6}}x^1 + \frac{1}{\sqrt{2}}x^2)}, e^{i(\frac{1}{\sqrt{6}}x^1 - \frac{1}{\sqrt{2}}x^2)}, e^{-i\sqrt{\frac{2}{3}}x^1} \right). \quad (16)$$

The corresponding covering  $\psi : \mathbb{R}^4 \rightarrow T \times \mathfrak{t}$  is

$$\psi(x, p) = \left( \varphi(x), \text{diag} \left( i(\frac{1}{\sqrt{6}}p_1 + \frac{1}{\sqrt{2}}p_2), i(\frac{1}{\sqrt{6}}p_1 - \frac{1}{\sqrt{2}}p_2), -i\sqrt{\frac{2}{3}}p_1 \right) \right). \quad (17)$$

**Remark 4.1.** Since  $\psi$  is a local diffeomorphism it is a local symplectomorphism and hence provides local Darboux coordinates on  $T \times \mathfrak{t}$ .

Now having constructed  $\psi$ , we can compose it with the map  $\lambda : T \times \mathfrak{t} \rightarrow \mathcal{P}$ , see (12), to obtain

$$\chi := \lambda \circ \psi : \mathbb{R}^4 \rightarrow \mathcal{P}. \quad (18)$$

Let  $\mathbb{R}_k^4 = \chi^{-1}(\mathcal{P}_k)$  denote the pre-image of the stratum  $\mathcal{P}_k$  under  $\chi$ ,  $k = 2, 1, 0$ . Using  $\mathbb{R}_k^4 = \psi^{-1}((T \times \mathfrak{t})_k)$  we find

$$\mathbb{R}_0^4 = \mathbb{R}_0^2 \times \{0\}, \quad \mathbb{R}_1^4 = \left( \bigcup_{j=1}^3 \bigcup_{l \in \mathbb{Z}} \mathbb{R}_{(j)l}^2 \times \mathbb{R}_{(j)0}^2 \right) \setminus \mathbb{R}_0^4, \quad \mathbb{R}_2^4 = \mathbb{R}^4 \setminus \mathbb{R}_1^4, \quad (19)$$

where

$$\begin{aligned} \mathbb{R}_0^2 &= \{ (l\sqrt{\frac{2}{3}}\pi, (l+2m)\sqrt{2}\pi) \mid l, m \in \mathbb{Z} \} \\ \mathbb{R}_{(1)l}^2 &= \{ (y, \sqrt{3}y + 2l\sqrt{2}\pi) \mid y \in \mathbb{R} \} \\ \mathbb{R}_{(2)l}^2 &= \{ (y, -\sqrt{3}y + 2l\sqrt{2}\pi) \mid y \in \mathbb{R} \} \\ \mathbb{R}_{(3)l}^2 &= \{ (y, l\sqrt{2}\pi) \mid y \in \mathbb{R} \}. \end{aligned}$$

The  $\mathbb{R}_{(j)l}^2$  are affine subspaces of  $\mathbb{R}^2$ , intersecting each other in the points of  $\mathbb{R}_0^2$ , see Figure 3. The  $\mathbb{R}_k^4$  are symplectic submanifolds of  $\mathbb{R}^4$ : for  $k = 0$  this is trivial, for  $k = 2$  it is obvious. For  $k = 1$  it follows from the fact that in the natural identification of  $T^*\mathbb{R}^2$  with  $\mathbb{R}^4$  utilized here,  $\mathbb{R}_{(j)l}^2 \times \mathbb{R}_{(j)l}^2$  corresponds to  $T^*\mathbb{R}_{(j)l}^2$ ,  $j = 1, 2, 3$ ,  $l \in \mathbb{Z}$ .

By restriction,  $\psi$  and  $\chi$  induce maps

$$\psi_k : \mathbb{R}_k^4 \rightarrow (T \times \mathfrak{t})_k, \quad \chi_k = \lambda_k \circ \psi_k : \mathbb{R}_k^4 \rightarrow \mathcal{P}_k, \quad k = 2, 1, 0, \quad (20)$$

respectively.

**Theorem 4.2.** *The map  $\chi$  is Poisson, i.e., for  $f, g \in C^\infty(\mathcal{P})$  there holds*

$$\chi^*\{f, g\}_{\mathcal{P}} = \frac{\partial(\chi^*f)}{\partial x^k} \frac{\partial(\chi^*g)}{\partial p_k} - \frac{\partial(\chi^*f)}{\partial p_k} \frac{\partial(\chi^*g)}{\partial x^k}.$$

*The maps  $\chi_k$  are local symplectomorphisms.*

*Proof.* This follows from Theorem 3.4. In addition, for the second assertion one has to use that the  $\psi_k$  are local symplectomorphisms. This is a consequence of the fact that  $(T \times \mathfrak{t})_k$  are embedded submanifolds of  $T \times \mathfrak{t}$ .  $\square$

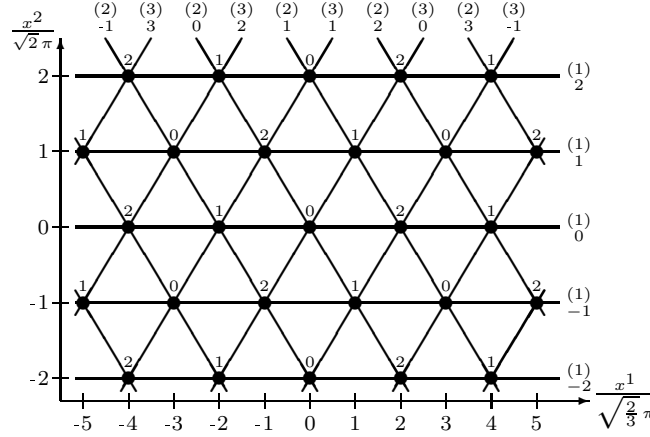


Figure 3: The subsets  $\mathbb{R}_0^2$  and  $\mathbb{R}_{(j)l}^2$  of  $\mathbb{R}^2$ . The elements of  $\mathbb{R}_0^2$  are represented by  $\bullet$  and are labelled by the element of  $\mathcal{X}_0$  they project to: 0, 1, 2 stands for  $\mathbb{1}$ ,  $\exp i\frac{2}{3}\pi\mathbb{1}$ ,  $\exp i\frac{4}{3}\pi\mathbb{1}$ , respectively. The affine subspaces  $\mathbb{R}_{(j)l}^2$  are labelled by  $\binom{j}{l}$ .

## 5 Description in terms of invariants

In this section, we derive the invariant-theoretic description of the reduced data of our model. Let us start with recalling the general theory. Consider an orthogonal representation of some Lie group  $H$  on a Euclidean space  $\mathbb{R}^n$ . The algebra of invariant polynomials of this representation is finitely generated [19]. Any finite set of generators  $\rho_1, \dots, \rho_p$  defines a map

$$\rho = (\rho_1, \dots, \rho_p) : \mathbb{R}^n / H \rightarrow \mathbb{R}^p.$$

This map is a homeomorphism onto its image [17] and the image is a closed semialgebraic subset of  $\mathbb{R}^p$ , i.e., it is the solution set of a logical combination of algebraic equations and inequalities. The equations are provided by the relations amongst the generators  $\rho_i$  and the inequalities keep track of their ranges. The set  $\{\rho_1, \dots, \rho_p\}$  and the map  $\rho$  are called a Hilbert basis and a Hilbert map for the representation, respectively. If  $V \subseteq \mathbb{R}^n$  is an  $H$ -invariant semialgebraic subset, then  $\rho$  restricts to a homeomorphism of  $V/H$  onto the image  $\rho(V) \subseteq \mathbb{R}^p$  and the image is again a semialgebraic subset. The equations are now given by the relations amongst the restricted mappings  $\rho_i|_V$  and the inequalities are given by their ranges.

### 5.1 Hilbert map

To apply the method explained above to our model, we consider the realification of the representation of  $G = \text{SU}(3)$  on  $M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$  by diagonal conjugation:

$$a \cdot (X_1, X_2) = (aX_1a^{-1}, aX_2a^{-1}) \quad (21)$$

and set  $V = J^{-1}(0) \subseteq G \times \mathfrak{g}$ . Indeed, since this (complex) representation is unitary w.r.t. the scalar product

$$\langle (X_1, X_2), (Y_1, Y_2) \rangle = \text{tr}(X_1^\dagger Y_1) + \text{tr}(X_2^\dagger Y_2), \quad (22)$$

the realification, equipped with the real part of (22) as a scalar product, is orthogonal. Moreover, the subset  $J^{-1}(0) \subseteq M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$  is defined by the equations

$$a^\dagger a = \mathbb{1}, \quad \det(a) = 1, \quad A^\dagger + A = 0, \quad aA - Aa = 0, \quad (23)$$

hence is real algebraic.

Since the invariant polynomials of the realification of a complex representation are given by the real and imaginary parts of the invariant polynomials of the original representation, we have to find the generators for the latter. According to [16], a set of generators for the invariant polynomials of the representation of  $SU(n)$  on  $M_n(\mathbb{C})^m$  by diagonal conjugation is given by the trace monomials up to order  $2^n - 1$  in  $X_1, \dots, X_m$  and  $X_1^\dagger, \dots, X_m^\dagger$ . The generators are subject to the relation

$$\sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) \prod_{\substack{(k_1, \dots, k_j) \\ \text{cycle of } \sigma}} \text{tr}(Y_{k_1} \cdots Y_{k_j}) = 0, \quad Y_1, \dots, Y_{n+1} \in M_n(\mathbb{C}), \quad (24)$$

called the fundamental trace identity (FTI). Thus, according to the general theory, the real and imaginary parts of the trace monomials up to order 7 in  $a, A$  and  $a^\dagger, A^\dagger$ , where  $(a, A) \in J^{-1}(0)$ , provide a homeomorphism of  $\mathcal{P}$  onto a semialgebraic subset of  $\mathbb{R}^p$  for some large  $p$ . However, for the restrictions of the trace monomials to  $J^{-1}(0)$  more relations hold than just the FTI. We can use them to reduce the set of generators and thus to simplify the Hilbert map. They arise from the matrix identities (23) and the Cayley-Hamilton theorem which says that the characteristic polynomial  $\chi_X$  of any  $X \in M_n(\mathbb{C})$  obeys  $\chi_X(X) = 0$ . The characteristic polynomials of  $a$  and  $A$  are

$$\chi_a(z) = -z^3 + \text{tr}(a)z^2 - \overline{\text{tr}(a)}z + 1, \quad \chi_A(z) = -z^3 + \frac{1}{2}\text{tr}(A^2)z + \frac{1}{3}\text{tr}(A^3), \quad (25)$$

respectively. Using (23), any trace monomial can be transformed to the form  $\text{tr}(a^k A^l)$  or its conjugate for some  $k, l$ . Using (25) it can then be rewritten as a polynomial in the monomials

$$\text{tr}(a), \quad \text{tr}(aA), \quad \text{tr}(aA^2), \quad \text{tr}(A^2), \quad \text{tr}(A^3).$$

We define

$$\begin{aligned} c_k &:= \text{Re}(\text{tr}(a(-iA)^k)), & d_k &:= \text{Im}(\text{tr}(a(-iA)^k)), & k &= 0, 1, 2, \\ t_k &:= \text{tr}((-iA)^k), & & & k &= 2, 3. \end{aligned}$$

As  $iA$  is self-adjoint,  $t_2$  and  $t_3$  are real. Thus, we arrive at the simplified Hilbert map

$$\rho_{\mathcal{P}} = (c_0, d_0, c_1, d_1, c_2, d_2, t_2, t_3) : \mathcal{P} \rightarrow \mathbb{R}^8.$$

By embedding  $G \hookrightarrow G \times \{0\} \subseteq J^{-1}(0)$ , from  $\rho_{\mathcal{P}}$  we obtain the Hilbert map for the action of  $G$  on itself by inner automorphisms, i.e., for the reduced configuration space  $\mathcal{X}$ :

$$\rho_{\mathcal{X}} = (c_0, d_0) : \mathcal{X} \rightarrow \mathbb{R}^2. \quad (26)$$

Analogously, embedding  $\mathfrak{g} \hookrightarrow \{\mathbb{1}\} \times \mathfrak{g} \subseteq J^{-1}(0)$  and using that on the image of this embedding there holds  $c_2 = t_2$  and  $c_1 = d_1 = d_2 = 0$ , we obtain the Hilbert map for the adjoint representation of  $\mathrm{SU}(3)$ , or the corresponding representation of  $S_3$  on  $\mathfrak{t}$ ,

$$\rho_{\mathrm{Ad}} = (t_2, t_3) : \mathrm{su}(3)/\mathrm{Ad} \rightarrow \mathbb{R}^2.$$

By construction, the maps  $\rho_{\mathcal{P}}$ ,  $\rho_{\mathcal{X}}$  and  $\rho_{\mathrm{Ad}}$  are homeomorphisms onto their images. The images will be denoted by  $\tilde{\mathcal{P}}$ ,  $\tilde{\mathcal{X}}$  and  $\tilde{\mathcal{Y}}$ , respectively. The images of the strata  $\mathcal{P}_k$  of  $\mathcal{P}$  and  $\mathcal{X}_k$  of  $\mathcal{X}$  will be denoted by  $\tilde{\mathcal{P}}_k$  and  $\tilde{\mathcal{X}}_k$ , respectively. As  $\tilde{\mathcal{P}}$ ,  $\tilde{\mathcal{X}}$  and  $\tilde{\mathcal{Y}}$  are projections of a semialgebraic subset, they are semialgebraic themselves. The reason why we consider  $\tilde{\mathcal{Y}}$  is that it will be needed in the discussion of  $\tilde{\mathcal{P}}$ .

## 5.2 Reduced configuration space and quotient of adjoint representation

The subset  $\tilde{\mathcal{X}}$  was discussed in [3]. We recall the results. A natural candidate for an inequality is given by the discriminant  $D(\chi_a)$  of  $\chi_a$ . Indeed, as  $a$  has eigenvalues  $\alpha, \beta$  and  $\overline{\alpha\beta}$ , where  $\alpha, \beta \in \mathrm{U}(1)$ ,

$$D(\chi_a) = (\alpha - \beta)^2(\alpha - \overline{\alpha\beta})^2(\beta - \overline{\alpha\beta})^2 = -|\alpha\beta|^2|\alpha - \beta|^2|\alpha - \overline{\alpha\beta}|^2|\beta - \overline{\alpha\beta}|^2 \leq 0.$$

Define

$$P_1(c_0(a), d_0(a)) := -D(\chi_a), \quad a \in \mathrm{SU}(3).$$

Expressing the discriminant in terms of the coefficients of  $\chi_a$ , see (25), yields

$$P_1(c_0, d_0) = 27 - c_0^4 - 2c_0^2d_0^2 - d_0^4 + 8c_0^3 - 24c_0d_0^2 - 18c_0^2 - 18d_0^2.$$

Moreover, define

$$P_0(c_0, d_0) := 9 - c_0^2 - d_0^2.$$

**Theorem 5.1.**  *$\tilde{\mathcal{X}}$  is the subset of  $\mathbb{R}^2$  defined by the inequality  $P_1 \geq 0$ . As subsets of  $\tilde{\mathcal{X}}$ , the strata are defined by the following equations and inequalities:*

$$\tilde{\mathcal{X}}_0: P_0 = 0, \quad \tilde{\mathcal{X}}_1: P_1 = 0 \text{ and } P_0 > 0, \quad \tilde{\mathcal{X}}_2: P_1 > 0.$$

*Proof.* By construction,  $\tilde{\mathcal{X}}$  is contained in the subset defined by  $P_1 \geq 0$ . The inverse inclusion was shown in [3]. To discuss the stratification, let  $a \in \mathcal{X}$  (again identified with  $\mathcal{A}$ ). One has  $a \in \mathcal{X}_2$  iff all its entries are distinct, i.e., iff  $D(\chi_a) \neq 0$ . This yields the assertion for  $\tilde{\mathcal{X}}_2$ . One has  $a \in \mathcal{X}_0$  iff all its entries are equal. This is equivalent to  $|tr(a)| = 3$ , i.e.,  $c_0(a)^2 + d_0^2(a) = 9$ , hence the assertion for  $\tilde{\mathcal{X}}_0$ . Then the assertion for  $\tilde{\mathcal{X}}_1$  follows.  $\square$

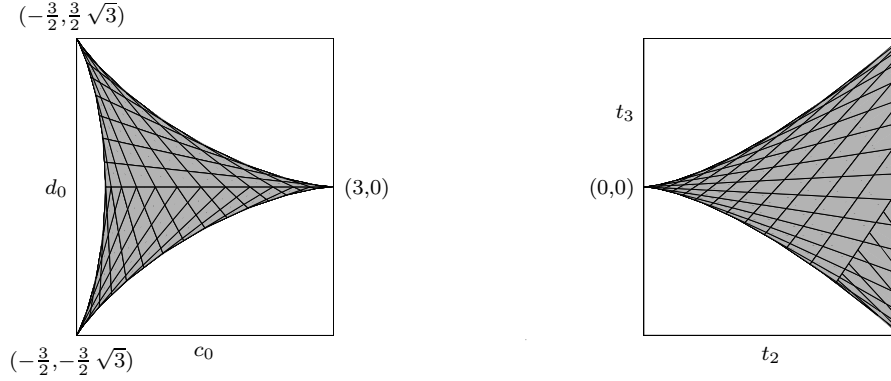


Figure 4: The subsets  $P_1 \geq 0$  (left) and  $P_2 \geq 0$  (right). The curve  $P_1 = 0$  is a 3-hypocycloid. All the singular points of the curves  $P_1 = 0$  and  $P_2 = 0$  are cusps.

The curve  $P_1 = 0$  is a 3-hypocycloid in a circle of radius 3 and  $\tilde{\mathcal{X}}$  is the subset of  $\mathbb{R}^2$  enclosed by this hypocycloid, see Figure 4.

Next, consider  $\tilde{\mathcal{Y}}$ . Again, the discriminant of  $\chi_A$  is a natural candidate for an inequality: as  $A$  has purely imaginary eigenvalues,  $D(\chi_A) \leq 0$ . Define

$$P_2(t_2(A), t_3(A)) := -D(\chi_A) \quad A \in \mathfrak{su}(3).$$

In terms of the coefficients of  $\chi_A$ ,

$$P_2(t_2, t_3) = \frac{1}{2}t_2^3 - 3t_3^2.$$

**Lemma 5.2.**  $\tilde{\mathcal{Y}}$  is the subset of  $\mathbb{R}^2$  defined by the inequality  $P_2 \geq 0$ . A matrix  $A \in \mathfrak{t}$  has  $n$  distinct entries iff the following conditions hold:

$$n = 1: \quad t_2 = 0, \quad n = 2: \quad P_2 = 0 \text{ and } t_2 > 0, \quad n = 3: \quad P_2 > 0.$$

*Proof.* By construction,  $\tilde{\mathcal{Y}}$  is contained in the subset of  $\mathbb{R}^2$  defined by  $P_2 \geq 0$ . Conversely, for any choice of  $(t_2, t_3) \in \mathbb{R}^2$  there exists  $A \in M_3(\mathbb{C})$  with these values for the invariants  $t_2, t_3$ . It may be chosen as a diagonal matrix with entries being the zeros of the polynomial  $\chi_A$ , see (25), where the traces have to be expressed in terms of the chosen values for  $t_2$  and  $t_3$ . It suffices to show that the inequality  $P_2(t_2, t_3) \geq 0$  implies  $A \in \mathfrak{g} = \mathfrak{su}(3)$ . Indeed, replacing  $z$  by  $iw$  yields  $i\chi_A = -w^3 + \frac{1}{2}t_2w + \frac{1}{3}t_3$ . This polynomial has real coefficients and discriminant  $-P_2(t_2, t_3) \leq 0$ . Therefore, its roots  $w_1, w_2, w_3$  are real. Since it does not contain a square term,  $w_1 + w_2 + w_3 = 0$ . Then  $A = \text{diag}(iw_1, iw_2, iw_3) \in \mathfrak{su}(3)$ .

The conditions that all entries are equal or that all entries are distinct are obvious. The condition that 2 entries are distinct then follows by observing that  $P_2 \geq 0$  implies  $t_2 \geq 0$ .  $\square$

The curve  $P_2 = 0$  is shown in Figure 4. The inequality  $P_2 \geq 0$  describes the part of the  $t_2$ - $t_3$ -plane to the right of this curve.

### 5.3 Reduced phase space

Now we turn to  $\tilde{\mathcal{P}}$ . First, let us look for equations defining  $J^{-1}(0)$  inside  $G \times \mathfrak{g}$ , i.e., reflecting the fact that  $a$  and  $A$  commute. The following 2 families of functions on  $G \times \mathfrak{g}$  obviously vanish on  $J^{-1}(0)$ :

$$\begin{aligned} f_k(a, A) &:= 2(-i)^{k-1} (\operatorname{tr}(A^k a A a^\dagger) - \operatorname{tr}(A^{k+1})) , \\ g_k(a, A) &:= (-i)^{k-1} (\operatorname{tr}(A^k a A a) - \operatorname{tr}(A^{k+1} a^2)) , \quad k = 1, 2, \dots \end{aligned}$$

The  $f_k$  and  $g_k$  are polynomials on  $G \times \mathfrak{g}$ . The  $f_k$  have real coefficients and the  $g_k$  have complex coefficients. Being invariant, they can be written as polynomials in the variables  $c_k, d_k, t_k$ . This way, we obtain 2 families of equations whose common zero set contains  $\rho(\mathcal{P})$ . They cannot all be independent. Indeed, for  $k \geq 3$ , using (25) one finds

$$\begin{aligned} f_k(a, A) &= -\frac{1}{2} \operatorname{tr}(A^2) f_{k-2}(a, A) + \frac{i}{3} \operatorname{tr}(A^3) f_{k-3}(a, A) , \\ g_k(a, A) &= -\frac{1}{2} \operatorname{tr}(A^2) g_{k-2}(a, A) + \frac{i}{3} \operatorname{tr}(A^3) g_{k-3}(a, A) , \end{aligned}$$

where  $f_0 = g_0 \equiv 0$ . Hence, the relevant equations are those arising from  $f_1, f_2, g_1$  and  $g_2$ . Taking the real and imaginary parts –  $f_1$  and  $f_2$  are already real – we obtain the following 6 equations:

$$f_1 = (3 + c_0^2 + d_0^2)t_2 - 2(c_1^2 + d_1^2) - 4(c_0c_2 + d_0d_2) = 0 , \quad (27)$$

$$f_2 = (3 - \frac{1}{3}(c_0^2 + d_0^2))t_3 - 2(c_1c_2 + d_1d_2) = 0 , \quad (28)$$

$$\operatorname{Re}(g_1) = c_0c_2 - d_0d_2 - 2c_0t_2 - c_1^2 + d_1^2 + 3c_2 = 0 , \quad (29)$$

$$\operatorname{Im}(g_1) = c_0d_2 + d_0c_2 + 2d_0t_2 - 2c_1d_1 - 3d_2 = 0 , \quad (30)$$

$$\operatorname{Re}(g_2) = \frac{1}{2}((c_0 + 1)c_1 - d_0d_1)t_2 + (\frac{1}{3}(c_0^2 - d_0^2) - c_0)t_3 - c_1c_2 + d_1d_2 = 0 , \quad (31)$$

$$\operatorname{Im}(g_2) = \frac{1}{2}((c_0 - 1)d_1 + d_0c_1)t_2 + (\frac{2}{3}c_0d_0 + d_0)t_3 - c_1d_2 - d_1c_2 = 0 . \quad (32)$$

These are the candidates for the equations defining  $\tilde{\mathcal{P}}$ .

Next, we look for the inequalities. Besides the two inequalities  $P_1 \geq 0$  and  $P_2 \geq 0$  found above, which contain only pure invariants, there is another obvious one which contains the mixed invariants  $c_2$  and  $d_2$ . Namely, for given  $a \in T$  and  $A \in \mathfrak{t}$ , the entries of  $a(-iA)^2$  are complex numbers whose modulus is given by the corresponding entry of  $(-iA)^2$ . Hence,  $|\operatorname{tr}(a(-iA)^2)| \leq \operatorname{tr}((-iA)^2)$ . In terms of the real invariants this reads  $P_3(c_2, d_2, t_2) \geq 0$ , where

$$P_3(c_2, d_2, t_2) := t_2^2 - c_2^2 - d_2^2 .$$

**Theorem 5.3.**  $\tilde{\mathcal{P}}$  is the subset of  $\mathbb{R}^8$  defined by the equations and inequalities

$$f_1 = f_2 = \operatorname{Re}(g_1) = \operatorname{Im}(g_1) = \operatorname{Im}(g_2) = 0 , \quad P_j \geq 0 , \quad j = 1, 2, 3 . \quad (33)$$

*Proof.* We have already checked that  $\tilde{\mathcal{P}}$  is contained in the subset (33). In order to prove the inverse inclusion, let there be given a point  $x = (c_0, d_0, c_1, d_1, c_2, d_2, t_2, t_3)$  from the subset (33). We have to show that there exists a pair  $(a, A) \in T \times \mathfrak{t}$  such that  $\rho_{\mathcal{P}}(a, A) = x$ .



Due to Theorem 5.1 and Lemma 5.2, there exist  $a \in T$  and  $A \in \mathfrak{t}$  with  $\rho_X(a) = (c_0, d_0)$  and  $\rho_{\text{Ad}}(A) = (t_2, t_3)$ , respectively. All pairs in the orbit of  $(a, A)$  under the direct product action of  $S_3 \times S_3$  on  $T \times \mathfrak{t}$  have the same values for the invariants  $c_0, d_0, t_2, t_3$ . Hence, if in (33) we view  $c_0, d_0, t_2, t_3$  as fixed parameters and  $c_1, d_1, c_2, d_2$  as the variables, it suffices to show that the number  $n_{\text{sol}}$  of distinct solutions of this system of equations and inequalities does not exceed the number  $n_{\text{orb}}$  of orbits under the diagonal action of  $S_3$  on the  $S_3 \times S_3$ -orbit of  $(a, A)$ . This holds in particular if  $n_{\text{sol}} = 1$ , i.e., if the solution is unique.

We start with separating  $c_2$  and  $d_2$  in the equations  $\text{Re}(g_1) = 0$  and  $\text{Im}(g_1) = 0$ :

$$P_0 c_2 = (3 - c_0) c_1^2 - (3 - c_0) d_1^2 - 2d_0 c_1 d_1 + 2(c_0(3 - c_0) + d_0^2) t_2, \quad (34)$$

$$P_0 d_2 = d_0 c_1^2 - d_0 d_1^2 - 2(3 + c_0) c_1 d_1 + 2d_0(3 + 2c_0) t_2. \quad (35)$$

The inequality  $P_1 \geq 0$  allows for 3 values of  $c_0, d_0$  where the factor  $P_0$  vanishes:

$$(c_0, d_0) = (3, 0), \left(-\frac{3}{2}, \frac{3}{2}\sqrt{3}\right), \left(-\frac{3}{2}, -\frac{3}{2}\sqrt{3}\right).$$

In the first case, the combination  $f_1 + 2\text{Re}(g_1) = 0$  yields  $c_1 = 0$ . Then (29) reads  $6(t_2 - c_2) + d_1^2 = 0$  and (30) reads  $d_1(t_2 - c_2) = 0$ . It follows  $d_1 = 0$  and  $c_2 = t_2$ . Then  $P_3 \geq 0$  implies  $d_2 = 0$ . In the other two cases,  $f_1 - 4\text{Re}(g_1) = 0$  implies  $c_1 = d_1 = 0$ . Resolving  $f_1$  for  $c_2$  and inserting this into  $P_3$  yields  $-(\sqrt{3}t_2 \pm 2d_2)^2 \geq 0$ . Hence,  $d_2 = \mp \frac{\sqrt{3}}{2}t_2$  and, then,  $c_2 = -\frac{1}{2}t_2$ . In all 3 cases  $n_{\text{sol}} = 1$ .

For the rest of the proof assume  $P_0 \neq 0$  (due to  $P_1 \geq 0$  then  $P_0 > 0$ ). Then  $c_2$  and  $d_2$  are fixed by (34) and (35) and can be replaced in (27) and (28):

$$2(9 + 6c_0 - 3c_0^2 + d_0^2) c_1^2 + 2Q_1 d_1^2 - 8d_0(3 + 2c_0) c_1 d_1 - P_1 t_2 = 0, \quad (36)$$

$$\begin{aligned} 2(c_0 - 3) c_1^3 + 2d_0 d_1^3 + 2d_0 c_1^2 d_1 + 2(9 + c_0) c_1 d_1^2 + 4(c_0^2 - 3c_0 - d_0^2) t_2 c_1 \\ - (12 + 8c_0) d_0 t_2 d_1 + \frac{1}{3} P_0^2 t_3 = 0, \end{aligned} \quad (37)$$

where we have introduced the notation

$$Q_1 = (3 - c_0)^2 - 3d_0^2.$$

The coefficient  $Q_1$  vanishes exactly for the 3 values of  $c_0, d_0$  which obey  $P_0 = 0$ . Hence, we can solve (36) for  $d_1$ ,

$$d_1 = \frac{1}{2Q_1} \left( (12 + 8c_0) d_0 c_1 \pm \sqrt{2P_1(Q_1 t_2 - 6c_1^2)} \right). \quad (38)$$

If  $t_2 = 0$  then  $c_1 = 0$ , because  $d_1$  must be real, and hence  $d_1 = 0$ . Due to  $P_2 \geq 0$ , also  $t_3 = 0$ . Then (34) and (35) imply  $c_2 = d_2 = 0$ . Thus, again  $n_{\text{sol}} = 1$ .

In the sequel assume  $t_2 \neq 0$  (due to  $P_2 \geq 0$  then  $t_2 > 0$ ). If  $P_1 = 0$ ,  $d_1$  is a multiple of  $c_1$ , hence replacing  $d_1$  in (37) yields a 3rd order polynomial equation which has at most 3 real solutions. I.e.,  $n_{\text{sol}} \leq 3$ . On the other hand, due to Theorem 5.1,  $a$  has 2 distinct entries. Due to Lemma 5.2,  $A$  has at least 2 distinct entries. Therefore,  $n_{\text{orb}} = 3$ .

In what follows we assume  $P_1(c_0, d_0) > 0$ . Then  $a$  has 3 distinct eigenvalues.

First, consider the case  $d_0 = 0$ . Here,  $d_1$  is a pure root and (37) contains  $d_1$  only in 2nd order. Hence, inserting (38) and discarding the global factor  $\frac{(c_0+3)^2}{3(c_0-3)}$  we obtain the 3rd order polynomial equation

$$24 c_1^3 - 3(3 - c_0)^2 t_2 c_1 - (3 - c_0)^3 t_3 = 0. \quad (39)$$

Since this equation has at most 3 real roots, each of which gives rise to at most 2 values of  $d_1$  by (38),  $n_{\text{sol}} \leq 6$ . It follows that in the case  $P_2 > 0$ , where  $A$  has 3 distinct eigenvalues,  $n_{\text{orb}} = 6 \geq n_{\text{sol}}$ . In the case  $P_2 = 0$ ,  $A$  has 2 distinct eigenvalues, so that  $n_{\text{orb}} = 3$ . To determine  $n_{\text{sol}}$  for this case, set

$$x := \sqrt[3]{\frac{t_3}{6}}.$$

Then  $t_2 = 6x^2$  and  $t_3 = 6x^3$ . Substituting this in (39) and dividing by 6 we obtain

$$4 c_1^3 - 3(3 - c_0)^2 x^2 c_1 - (3 - c_0)^3 x^3 = 0.$$

Since  $x \neq 0$  by assumption, the solutions of this equation are given by  $c_1 = \tilde{c}_1 x$ , where  $\tilde{c}_1$  are the solutions of the same equation with  $x = 1$ . We find  $\tilde{c}_1 = 3 - c_0$  with multiplicity 1 and  $\tilde{c}_1 = \frac{1}{2}(c_0 - 3)$  with multiplicity 2. Then (38) yields  $d_1 = 0$  in the first case and  $d_1 = \pm \frac{3}{2} \sqrt{(3 - c_0)(1 + c_0)} x$  in the second one. Thus,  $n_{\text{sol}} = 3 = n_{\text{orb}}$ .

Next, consider the case  $d_0 \neq 0$ . We insert (38) into (37) and write this equation in the form

$$\pm 3d_0(Q_1 t_2 - 24 c_1^2) \sqrt{2 P_1(Q_1 t_2 - 6 c_1^2)} = Q, \quad (40)$$

where  $Q$  is some polynomial and we have omitted a common factor  $3\sqrt{2}P_0^2/Q_1^3$  to avoid fractures. By squaring (40) we obtain the 6th order polynomial equation in  $c_1$

$$1152 c_1^6 - 288 Q_1 t_2 c_1^4 + 96 Q_2 t_3 c_1^3 + 18 Q_1^2 t_2^2 c_1^2 - 12 Q_3 t_2 t_3 c_1 + 2 Q_1^3 t_3^2 - 9 P_1 d_0^2 t_2^3 = 0, \quad (41)$$

where

$$\begin{aligned} Q_2 &= c_0^3 + 9c_0 d_0^2 - 9c_0^2 + 27d_0^2 + 27c_0 - 27, \\ Q_3 &= c_0^5 + 6c_0^3 d_0^2 - 27c_0 d_0^4 - 15c_0^4 - 81d_0^4 + 90c_0^3 - 162c_0 d_0^2 \\ &\quad - 270c_0^2 + 324d_0^2 + 405c_0 - 243, \end{aligned}$$

and we have omitted a global factor  $Q_1^3$ . To a solution  $c_1$  of (41) for which the l.h.s. of (40) does not vanish there corresponds one of the two signs in (40) and hence by (38) a unique value for  $d_1$ . To a solution for which  $Q_1 t_2 - 6 c_1^2 = 0$  there corresponds a unique  $d_1$  anyhow. To a solution for which  $Q_1 t_2 - 24 c_1^2 = 0$  there correspond two values of  $d_1$ , but such a solution necessarily has multiplicity 2. (This phenomenon should be interpreted the other way around: generically (41) has distinct solutions  $c_1$ , each with its own associated  $d_1$ . When 2 of the solutions happen to coincide, the associated values of  $d_1$  seem to emerge from the same  $c_1$ .) From these observations we conclude that  $n_{\text{sol}} \leq 6$ , so that for  $P_2 > 0$  we have  $n_{\text{orb}} = 6 \geq n_{\text{sol}}$ .

It remains to consider the case  $P_2 = 0$ , where  $n_{\text{orb}} = 3$ . As before, we replace  $t_2 = 6x^2$  and  $t_3 = 6x^3$  in (41) and argue that the solutions of the resulting equation are given by  $c_1 = \tilde{c}_1 x$ , where  $\tilde{c}_1$  are the solutions of this equation with  $x$  set to 1. The latter equation turns out to be the square of

$$4\tilde{c}_1^3 - 3Q_1\tilde{c}_1 + Q_2 = 0, \quad (42)$$

hence it has at most 3 distinct real solutions  $\tilde{c}_1$ . We claim that for none of the corresponding solutions  $c_1 = \tilde{c}_1 x$  the factor  $Q_1 t_2 - 24c_1^2 = 6x(Q_1 - 4\tilde{c}_1^2)$  in (40) vanishes. Assume, on the contrary,  $Q_1 - 4\tilde{c}_1^2 = 0$ . Inserting  $\tilde{c}_1 = \pm\sqrt{Q_1}$  into (42) and separating the terms with the root yields  $\pm Q_1 \sqrt{Q_1} = Q_2$ . Taking the square we obtain  $27d_0^2 P_1 = 0$ , in contradiction to the assumptions  $d_0 \neq 0$  and  $P_1 \neq 0$ . It follows that to each  $c_1$  there corresponds a unique value for  $d_1$ . Thus,  $n_{\text{sol}} = 3 = n_{\text{orb}}$ .

This completes the proof of Theorem 5.3.  $\square$

**Remark 5.4.** As a by-product of the proof we have seen that the 6 invariants  $c_0, d_0, c_1, d_1, t_2, t_3$  are sufficient to separate the points of  $\mathcal{P}$ . Hence, they define a homeomorphism of  $\mathcal{P}$  onto the projection of  $\tilde{\mathcal{P}}$  to  $\mathbb{R}^6$ . (Outside some 'momentum cutoff'  $\|A\| \leq k$  the homeomorphism property is obvious and inside one uses that a bijection of a compact space onto a Hausdorff space is a homeomorphism.) The invariants  $c_2, d_2$  cannot be expressed as polynomials in the other invariants, though. However, according to (34) and (35) and the subsequent discussion, on  $\tilde{\mathcal{P}}$  they can be expressed as continuous functions in the other invariants. For  $(c_0, d_0) \neq (3, 0), (-\frac{3}{2}, \pm\frac{3}{2}\sqrt{3})$ ,

$$c_2 = P_0^{-1} \left( (3 - c_0) c_1^2 - (3 - c_0) d_1^2 - 2d_0 c_1 d_1 + 2(c_0(3 - c_0) + d_0^2) t_2 \right), \quad (43)$$

$$d_2 = P_0^{-1} \left( d_0 c_1^2 - d_0 d_1^2 - 2(3 + c_0) c_1 d_1 + 2d_0(3 + 2c_0) t_2 \right), \quad (44)$$

whereas for  $(c_0, d_0) = (3, 0), (-\frac{3}{2}, \pm\frac{3}{2}\sqrt{3})$ , in the respective order,

$$(c_2, d_2) = (3t_2, 0), \left(-\frac{1}{2}t_2, \mp\frac{\sqrt{3}}{2}t_2\right). \quad (45)$$

One can extend  $c_2$  and  $d_2$  to rational functions on  $\mathbb{R}^6$  by means of the expressions on the r.h.s. of (43) and (44). Then the values (45) have to be understood as the limits when  $(c_0, d_0) \rightarrow (3, 0), (-\frac{3}{2}, \pm\frac{3}{2}\sqrt{3})$  along  $\tilde{\mathcal{P}}$ .

On the level of the semialgebraic sets, the projection  $\tilde{\pi} : \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{X}}$  is just given by the natural projection to the  $c_0$ - $d_0$  plane. Figure 5 shows the projections of the fibres  $\tilde{\pi}^{-1}(c_0, d_0)$  to the 3-dimensional subspace spanned by the coordinates  $t_2, t_3$  and  $c_1$  for 5 different points  $(c_0, d_0) \in \tilde{\mathcal{X}}$ . In addition, the projection of the fibres to the  $t_2$ - $t_3$  plane, which just coincides with  $\tilde{\mathcal{Y}}$ , is shown, too. The figures were drawn using a parametrization of the invariants, induced by a parametrization of the matrices  $a$  and  $A$ .

For  $(c_0, d_0)$  belonging to the stratum  $\tilde{\mathcal{X}}_2$  the fibre  $\tilde{\pi}^{-1}(c_0, d_0)$  is a full 2-plane, folded 3 times over the curve  $P_2 = 0$  (Figures (a) and (b)). The self-intersections present in these

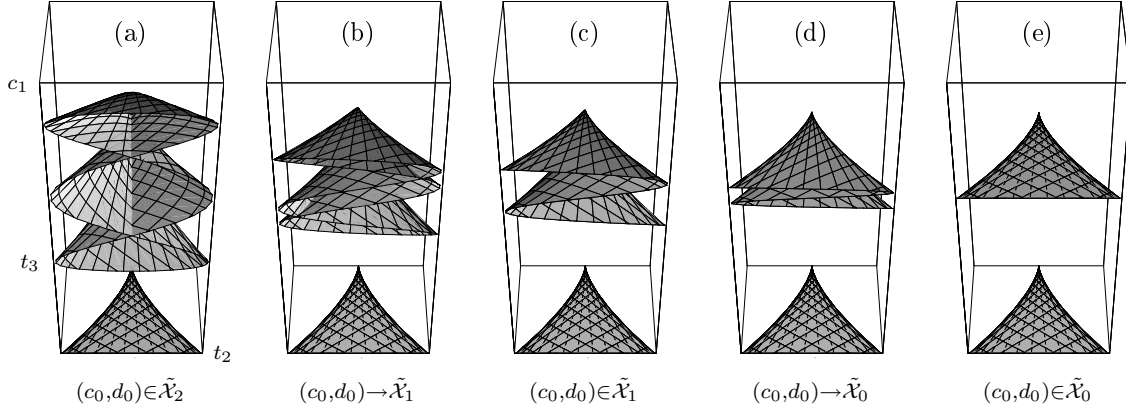


Figure 5: Projection of the fibres  $\tilde{\pi}^{-1}(c_0, d_0)$  to the  $t_2$ - $t_3$ - $c_1$  plane (top) and the  $t_2$ - $t_3$  plane (bottom).

figures are remnants of the projection to the  $t_2$ - $t_3$ - $c_1$  hyperplane. In fact, they correspond to solutions  $c_1$  of (41) of multiplicity 2 where the factor  $Q_1 t_2 - 24c_1^2$  in (40) vanishes, so that both values of  $d_1$  in (38) are allowed. Since the latter are distinct (unless  $t_2 = 0$ ), the fictitious self-intersection of the fibre is not present in the full  $\mathbb{R}^8$ .

When  $(c_0, d_0)$  approaches the stratum  $\tilde{\mathcal{X}}_1$ , i.e., the curve  $P_1 = 0$ , the two halves of the plane come closer (Figure (b)). For  $(c_0, d_0) \in \tilde{\mathcal{X}}_1$  they meet each other and thus make the fibre a half-plane with a double fold (Figure (c)). When moving  $(c_0, d_0)$  further along  $\tilde{\mathcal{X}}_1$  towards one of the points of the stratum  $\tilde{\mathcal{X}}_0$  the three layers of this half-plane approach each other (Figure (d)) to finally merge to a 'sixth-plane' cone for  $(c_0, d_0) \in \tilde{\mathcal{X}}_0$  (Figure (e)). This illustrates the abstract description of the fibres in Section 3.4.

## 5.4 Stratification

We determine the equations and inequalities defining the strata of  $\tilde{\mathcal{P}}$ . We will make use of the discriminant of the characteristic polynomial  $\chi_{aA}$ . Define

$$P_4(c_0(a, A), \dots, t_3(a, A)) := \text{Re}(D(\chi_{aA})).$$

Using (24) and (25) one finds

$$\chi_{aA}(z) = -z^3 + \text{tr}(aA) z^2 + \left( \frac{1}{2} \text{tr}(A^2) \overline{\text{tr}(a)} - \overline{\text{tr}(aA^2)} \right) z + \frac{1}{3} \text{tr}(A^3).$$

It follows

$$\begin{aligned} P_4 = & c_1^2 c_2^2 - c_1^2 d_2^2 + 4 c_1 d_1 c_2 d_2 + d_1^2 d_2^2 - \frac{4}{3} t_3 c_1^3 - c_0 t_2 c_1^2 c_2 + d_0 t_2 c_1^2 d_2 + 4 t_3 c_1 d_1^2 \\ & - 2 d_0 t_2 c_1 d_1 c_2 - 2 c_0 t_2 c_1 d_1 d_2 + c_0 t_2 d_1^2 c_2 - d_0 t_2 d_1^2 d_2 - 4 c_2^3 \\ & + 12 c_2 d_2^2 + \frac{1}{4} (c_0^2 - d_0^2) t_2^2 c_1^2 + c_0 d_0 t_2^2 c_1 d_1 + 6 t_3 c_1 c_2 - \frac{1}{4} (c_0^2 - d_0^2) t_2^2 d_1^2 \\ & + 6 t_3 d_1 d_2 + 6 c_0 t_2 c_2^2 - 12 d_0 t_2 c_2 d_2 - 6 c_0 t_2 d_2^2 - 3 c_0 t_2 t_3 c_1 \\ & - 3 d_0 t_2 t_3 d_1 + 3 (d_0^2 - c_0^2) t_2^2 c_2 + 6 c_0 d_0 t_2^2 d_2 + \frac{1}{2} c_0 (c_0^2 - 3 d_0^2) t_2^3 - 3 t_3^2 \end{aligned}$$

**Theorem 5.5.** *As subsets of  $\tilde{\mathcal{P}}$ , the strata  $\tilde{\mathcal{P}}_k$  are defined by the following equations and inequalities:*

$$\begin{aligned}\tilde{\mathcal{P}}_0 &: P_0 = 0 \text{ and } t_2 = 0 \\ \tilde{\mathcal{P}}_1 &: P_1 = P_2 = P_4 = 0 \text{ and } (P_0 > 0 \text{ or } t_2 > 0) \\ \tilde{\mathcal{P}}_2 &: P_1 > 0 \text{ or } P_2 > 0 \text{ or } P_4 \neq 0\end{aligned}$$

*Proof.* Let  $(a, A) \in T \times \mathfrak{t}$  be given.

The pair  $(a, A)$  is invariant under the full  $S_3$ -action iff so are  $a$  and  $A$  individually. According to Theorem 5.1 and Lemma 5.2, this holds iff  $P_0 = 0$  and  $t_2 = 0$ , respectively. Next, assume that  $(a, A)$  has nontrivial stabilizer. Then there are 2 entries which coincide for  $a$  and  $A$  simultaneously. Then  $aA$  has a degenerate eigenvalue. It follows  $D(\chi_{aA}) = 0$  and, hence,  $P_4 = 0$ . Conversely, assume  $P_1 = P_2 = P_4 = 0$ . Then  $a$  and  $A$  both have coinciding entries. Up to  $S_3$ -action we can assume  $a = \text{diag}(\alpha, \alpha, \bar{\alpha}^2)$ ,  $\alpha \in \text{U}(1)$ . Then  $A$  can be

$$\text{diag}(ix, ix, -2ix), \quad \text{diag}(ix, -2ix, ix) \quad \text{or} \quad \text{diag}(-2ix, ix, ix), \quad x \in \mathbb{R}. \quad (46)$$

If  $x = 0$  or  $\alpha^3 = 1$  then in all 3 cases  $(a, A)$  has nontrivial stabilizer. Hence, assume  $x \neq 0$  and  $\alpha^3 \neq 1$ . In the second and the third case,

$$D(\chi_{aA}) = (\alpha x + 2\alpha x)^2(\alpha x - \bar{\alpha}^2 x)^2(2\alpha x + \bar{\alpha}^2 x)^2 = 9x^6(2\alpha^3 - \bar{\alpha}^3 - 1)^2$$

Taking the real part and replacing  $\text{Im}(\alpha^3)^2 = 1 - \text{Re}(\alpha^3)^2$  yields

$$P_4 = 72x^6(\text{Re}(\alpha^3) - 1)^2 = 0.$$

Hence,  $x = 0$  or  $\alpha^3 = 1$ , in contradiction to the assumption. Therefore,  $A = \text{diag}(ix, ix, -2ix)$  and hence  $(a, A)$  has nontrivial stabilizer. This yields the equations for  $\tilde{\mathcal{P}}_1$ . The inequalities for  $\tilde{\mathcal{P}}_1$  and  $\tilde{\mathcal{P}}_2$  are obvious.  $\square$

## 5.5 Poisson structure

The brackets of the generating invariants  $c_0, \dots, t_3$ , taken in the Poisson algebra  $C^\infty(\mathcal{P})$ , define a Poisson structure on  $\mathbb{R}^8$  by

$$\{f, g\} := \sum_{i,j=1}^8 \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \{x_i, x_j\}, \quad (47)$$

where  $(x_1, \dots, x_8) = (c_0, \dots, t_3)$ . This Poisson structure rules the dynamics on  $\tilde{\mathcal{P}}$ , see the brief remark in Section 6. The Poisson brackets in  $C^\infty(\mathcal{P})$  are defined by

$$\{f, g\} = \omega(X_f, X_g), \quad f, g \in C^\infty(G \times \mathfrak{g}),$$

where the symplectic form  $\omega$  is given by (9) and  $X_f, X_g$  are the Hamiltonian vector fields associated with  $f$  and  $g$ , respectively. They are defined pointwise by

$$\omega_{(a,A)}(X_f, X) = -X(f), \quad (48)$$

for all  $X \in T_{(a,A)}(G \times \mathfrak{g})$  and  $(a, A) \in G \times \mathfrak{g}$ . Here  $X(f)$  is the directional derivative of  $f$  along  $X$ . As in Subsection 3.3 we write the tangent vectors in the form

$$(X_f)_{(a,A)} = (R'_a B_f, (A, C_f)), \quad X = (R'_a B, (A, C))$$

with  $B_f, C_f, B, C \in \mathfrak{g}$ . Although it is not indicated by the notation,  $B_f$  and  $C_f$  depend on  $a$  and  $A$ , i.e., they are  $\mathfrak{g}$ -valued functions on  $G \times \mathfrak{g}$ . Using (9) and the invariance of the scalar product  $\langle \cdot, \cdot \rangle$  to rewrite the l.h.s. of (48), and using the curve  $(\exp(tB)a, A + tC)$  to represent  $X$ , (48) becomes

$$\langle B_f, C \rangle + \langle [B_f, A] - C_f, B \rangle = - \left. \frac{d}{dt} \right|_{t=0} f((\exp(tB)a, A + tC)), \quad \forall B, C \in \mathfrak{g}. \quad (49)$$

Putting  $B = 0$  yields  $B_f$ , then putting  $C = 0$  and replacing  $B_f$  in the commutator yields  $C_f$ . Having found the Hamiltonian vector fields associated with the invariants this way, the Poisson brackets are then given pointwise by

$$\{f, g\}((a, A)) = \langle B_f, C_g \rangle - \langle C_f, B_g \rangle - \langle A, [B_f, B_g] \rangle \quad (50)$$

Since it suffices to compute the brackets on the level set  $J^{-1}(0)$ , we may always assume  $(a, A) \in J^{-1}(0)$ . This simplifies the computations considerably. In particular, the commutators in (49) and (50) happen to vanish.

Let us illustrate the calculation by the bracket  $\{c_1, d_1\}$ . For  $c_1$  and  $d_1$ , (49) reads

$$\begin{aligned} \langle B_{c_1}, C \rangle + \langle [B_{c_1}, A] - C_{c_1}, B \rangle &= -\text{Im} \langle a, C \rangle - \text{Im} \langle aA, B \rangle \\ \langle B_{d_1}, C \rangle + \langle [B_{d_1}, A] - C_{d_1}, B \rangle &= -\text{Re} \langle a, C \rangle - \text{Re} \langle aA, B \rangle \end{aligned}$$

To express the r.h.s. in terms of scalar products of  $B$  and  $C$  with elements of  $\mathfrak{g}$ , let  $\Pi_+$  and  $\Pi_-$  denote the projections of  $M_3(\mathbb{C})$  onto the traceless Hermitian and traceless anti-Hermitian matrices, respectively. I.e.,

$$\Pi_{\pm}(D) = \frac{1}{2}(D \pm D^\dagger) - \frac{1}{6}(\text{tr}(D) \pm \overline{\text{tr}(D)}), \quad D \in M_3(\mathbb{C}).$$

Both  $\Pi_-$  and  $i\Pi_+$  map  $M_3(\mathbb{C})$  to  $\mathfrak{g}$  and for any  $D \in M_3(\mathbb{C})$  and  $B \in \mathfrak{g}$  one has

$$\text{Re} \langle D, B \rangle = \langle \Pi_-(D), B \rangle, \quad \text{Im} \langle D, B \rangle = \langle i\Pi_+(D), B \rangle. \quad (51)$$

This way, we obtain the Hamiltonian vector fields of the invariants:

$$\begin{aligned} B_{c_0} &= 0, & C_{c_0} &= -\Pi_-(a) = -\frac{1}{2}(a - a^\dagger) + \frac{i}{3}d_0, \\ B_{d_0} &= 0, & C_{d_0} &= i\Pi_+(a) = \frac{i}{2}(a + a^\dagger) - \frac{i}{3}c_0, \\ B_{c_1} &= -i\Pi_+(a) = -\frac{i}{2}(a + a^\dagger) + \frac{i}{3}c_0, & C_{c_1} &= i\Pi_+(aA) = \frac{i}{2}(a - a^\dagger)A + \frac{i}{3}d_1, \\ B_{d_1} &= -\Pi_-(a) = -\frac{1}{2}(a - a^\dagger) + \frac{i}{3}d_0, & C_{d_1} &= \Pi_-(aA) = \frac{1}{2}(a + a^\dagger)A - \frac{i}{3}c_1, \\ B_{c_2} &= -2\Pi_-(aA) = -(a + a^\dagger)A + \frac{2i}{3}c_1, & C_{c_2} &= \Pi_-(aA^2) = \frac{1}{2}(a - a^\dagger)A^2 + \frac{i}{3}d_2, \\ B_{d_2} &= 2i\Pi_+(aA) = i(a - a^\dagger)A + \frac{2i}{3}d_1, & C_{d_2} &= -i\Pi_+(aA^2) = -\frac{i}{2}(a + a^\dagger)A^2 - \frac{i}{3}c_2, \\ B_{t_2} &= -2A, & C_{t_2} &= 0, \\ B_{t_3} &= 3i\Pi_+(A^2) = 3iA^2 + it_2, & C_{t_3} &= 0. \end{aligned}$$

There hold the relations  $B_{c_1} = -C_{d_0}$ ,  $B_{d_1} = -C_{c_0}$ ,  $B_{c_2} = -2C_{d_1}$ ,  $B_{d_2} = 2C_{c_1}$ . According to (50), e.g.,

$$\begin{aligned}\{c_1, d_1\} &= \langle B_{c_1}, C_{d_1} \rangle - \langle C_{c_1}, B_{d_1} \rangle = \langle -i\Pi_+(a), C_{d_1} \rangle - \langle i\Pi_+(aA), B_{d_1} \rangle \\ &= -\operatorname{Im}\langle a, C_{d_1} \rangle - \operatorname{Im}\langle aA, B_{d_1} \rangle.\end{aligned}$$

By replacing  $C_{d_1}$  and  $B_{c_1}$  using the above explicit expressions and rewriting the resulting scalar products in terms of the invariants  $c_0, \dots, t_3$  we finally arrive at the desired Poisson brackets:

$$\begin{aligned}\{c_0, d_0\} &= 0, & \{c_1, d_1\} &= \frac{1}{3}(c_0c_1 + d_0d_1) \\ \{t_2, t_3\} &= 0, & \{c_2, d_2\} &= -2t_3 + \frac{2}{3}(c_1c_2 + d_1d_2) \\ \{c_0, c_1\} &= -\frac{2}{3}c_0d_0 - d_0, & \{d_0, d_1\} &= \frac{2}{3}c_0d_0 + d_0 \\ \{c_0, d_1\} &= \frac{1}{2}c_0^2 - \frac{1}{6}d_0^2 - c_0 - \frac{3}{2}, & \{d_0, c_1\} &= \frac{1}{6}c_0^2 - \frac{1}{2}d_0^2 - c_0 + \frac{3}{2} \\ \{c_0, c_2\} &= -c_0d_1 - \frac{1}{3}d_0c_1 + d_1, & \{d_0, d_2\} &= \frac{1}{3}c_0d_1 + d_0c_1 - d_1 \\ \{c_0, d_2\} &= c_0c_1 - \frac{1}{3}d_0d_1 + c_1, & \{d_0, c_2\} &= \frac{1}{3}c_0c_1 - d_0d_1 + c_1 \\ \{c_1, c_2\} &= -\frac{5}{6}c_0d_2 - \frac{1}{2}d_0c_2 - \frac{1}{2}d_0t_2 + \frac{1}{2}d_2 + \frac{2}{3}c_1d_1 \\ \{c_1, d_2\} &= \frac{5}{6}c_0c_2 - \frac{1}{2}d_0d_2 - \frac{1}{2}c_0t_2 - \frac{3}{2}t_2 + \frac{1}{2}c_2 + \frac{2}{3}d_1^2 \\ \{d_1, c_2\} &= \frac{1}{2}c_0c_2 - \frac{5}{6}d_0d_2 - \frac{1}{2}c_0t_2 + \frac{1}{2}c_2 + \frac{3}{2}t_2 - \frac{2}{3}c_1^2 \\ \{d_1, d_2\} &= \frac{1}{2}c_0d_2 + \frac{5}{6}d_0c_2 + \frac{1}{2}d_0t_2 - \frac{1}{2}d_2 - \frac{2}{3}c_1d_1 \\ \{c_0, t_2\} &= -2d_1, & \{d_0, t_2\} &= 2c_1 \\ \{c_1, t_2\} &= -2d_2, & \{d_1, t_2\} &= 2c_2 \\ \{c_2, t_2\} &= -t_2d_1 - \frac{2}{3}t_3d_0, & \{d_2, t_2\} &= t_2c_1 + \frac{2}{3}t_3c_0 \\ \{c_0, t_3\} &= t_2d_0 - 3d_2, & \{d_0, t_3\} &= -t_2c_0 + 3c_2 \\ \{c_1, t_3\} &= -\frac{1}{2}t_2d_1 - t_3d_0, & \{d_1, t_3\} &= \frac{1}{2}t_2c_1 + t_3c_0 \\ \{c_2, t_3\} &= -\frac{1}{2}t_2d_2 - t_3d_1, & \{d_2, t_3\} &= \frac{1}{2}t_2c_2 + t_3c_1\end{aligned}\tag{52}$$

**Remark 5.6.** Another description of the reduced phase space in terms of invariants can be constructed as follows [7, 8]. The polar map  $(a, A) \mapsto a \exp(-iA)$  yields a diffeomorphism of  $T \times \mathfrak{t}$  onto the complexification  $T^{\mathbb{C}}$ , which is isomorphic to the direct product of two copies of the group of nonzero complex numbers. This diffeomorphism passes to an isomorphism of stratified symplectic space from  $\mathcal{P}$  onto  $T^{\mathbb{C}}/S_3$ . The real invariants for the latter quotient are the elementary bisymmetric functions on  $T^{\mathbb{C}}$ , obtained from the elementary symmetric functions by bilinearization w.r.t. the holomorphic coordinates and their complex conjugates. This description is the starting point for stratified Kähler quantization in [6, 7]. It also has the great advantage that it directly generalizes to  $SU(n)$  and further to an arbitrary compact Lie group. For classical dynamics, however, it has the drawback that the kinetic energy is not polynomial in the generating invariants.

## 6 Towards classical dynamics (an outlook)

In this final section, we make some general remarks on the dynamics on  $\mathcal{P}$  and  $\mathcal{X}$ . A detailed study will be carried out in a subsequent paper.

In terms of the symplectic covering  $\chi$  of Section 4, the dynamics can be described as follows. Given a Hamiltonian function  $H \in C^\infty(\mathcal{P})$ , the lift  $\chi^*H$  is a Hamiltonian function on  $\mathbb{R}^4$ . Let the curve  $(x(t), p(t))$  be a solution of the Hamiltonian equations associated with  $\chi^*H$ ,

$$\dot{p}_j = -\frac{\partial(\chi^*H)}{\partial x^j}, \quad \dot{x}^j = \frac{\partial(\chi^*H)}{\partial p_j}, \quad j = 1, 2. \quad (53)$$

To be a solution is a local property. Since the map  $\psi : \mathbb{R}^4 \rightarrow T \times \mathfrak{t}$  is a local symplectomorphism, then  $\psi((x(t), y(t)))$  is a solution of the Hamiltonian equations of  $\lambda^*H$  on  $T \times \mathfrak{t}$ . According to point 2 of Remark 3.1, this curve stays inside  $(T \times \mathfrak{t})_k$  for some  $k = 2, 1, 0$ . Hence,  $(x(t), p(t))$  stays inside the corresponding  $\mathbb{R}_k^4$  and  $\chi((x(t), p(t))) = \chi_k((x(t), p(t)))$  is a curve in  $\mathcal{P}_k$ . Since  $\chi_k$  is a local symplectomorphism by Theorem 4.2, then this curve is a solution of the Hamiltonian equations of the Hamiltonian function  $H|_{\mathcal{P}_k}$  (restriction) on the stratum  $\mathcal{P}_k$ . This way, the Hamiltonian dynamics on  $\mathcal{P}$  w.r.t.  $H$  is completely solved by the Hamiltonian dynamics w.r.t.  $\chi^*H$  on  $\mathbb{R}^4$ . Furthermore, the trajectories in  $\mathcal{X}$  are given by  $\pi \circ \chi((x(t), p(t)))$ . Define  $\tilde{\chi} : \mathbb{R}^2 \rightarrow \mathcal{X}$  to be the composition of the covering  $\varphi : \mathbb{R}^2 \rightarrow T$ , see (16), with the natural projection  $T \rightarrow \mathcal{X}$ . Then  $\pi \circ \chi((x(t), p(t))) = \tilde{\chi}(x(t))$ . Hence, for the discussion of the trajectories in  $\mathcal{X}$ , it suffices to consider the trajectories  $x(t)$  in  $\mathbb{R}^2$ . An explicit realization of the trajectories in  $\mathcal{X}$  can be obtained by passing to  $\mathbb{R}^2$  by means of the Hilbert map  $\rho_{\mathcal{X}}$ , see (26). One finds

$$\begin{aligned} \rho_{\mathcal{X}} \circ \tilde{\chi}(x(t)) = & \left( \cos\left(\frac{1}{\sqrt{6}}x^1(t) + \frac{1}{\sqrt{2}}x^2(t)\right) + \cos\left(\frac{1}{\sqrt{6}}x^1(t) - \frac{1}{\sqrt{2}}x^2(t)\right) + \cos\left(\sqrt{\frac{2}{3}}x^1(t)\right), \right. \\ & \left. \sin\left(\frac{1}{\sqrt{6}}x^1(t) + \frac{1}{\sqrt{2}}x^2(t)\right) + \sin\left(\frac{1}{\sqrt{6}}x^1(t) - \frac{1}{\sqrt{2}}x^2(t)\right) - \sin\left(\sqrt{\frac{2}{3}}x^1(t)\right) \right). \end{aligned}$$

Now consider the Hamiltonian (2). One has

$$\chi^*H = \frac{\delta^3}{2}(p_1^2 + p_2^2) + \frac{1}{g^2\delta} \left( 3 - \cos\left(\frac{x^1}{\sqrt{6}} + \frac{x^2}{\sqrt{2}}\right) - \cos\left(\frac{x^1}{\sqrt{6}} - \frac{x^2}{\sqrt{2}}\right) - \cos\left(\sqrt{\frac{2}{3}}x^1\right) \right) \quad (54)$$

The Hamiltonian has the standard structure, consisting of a kinetic energy term and a potential term. The potential is represented in Figure 6. Its minimal value is 0, it is taken at the points

$$(3l\sqrt{\frac{2}{3}}\pi, (3l+2m)\sqrt{2}\pi) \in \mathbb{R}_0^2, \quad l, m \in \mathbb{Z}.$$

The maximal value is  $\frac{1}{g^2\delta}\frac{9}{2}$ , taken at

$$((3l+1)\frac{2}{\sqrt{3}}\pi, (3l+2m+1)\sqrt{2}\pi), ((3l+2)\frac{2}{\sqrt{3}}\pi, (3l+2m+2)\sqrt{2}\pi) \in \mathbb{R}_0^2, \quad l, m \in \mathbb{Z}.$$

In addition, the potential has saddle points at

$$(3l\sqrt{\frac{2}{3}}\pi, (3l+2m+1)\sqrt{2}\pi) \in \mathbb{R}_1^2, \quad l, m \in \mathbb{Z}.$$



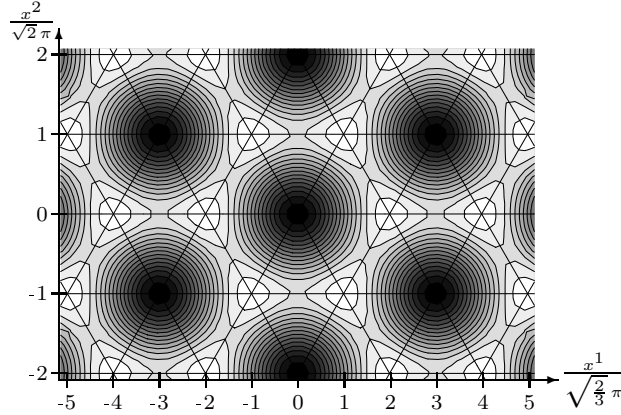


Figure 6: Level diagram of the potential of the Hamiltonian (54). Dark regions mean low potential.

In the representation of  $\mathbb{R}^2$  in Figure 3, the minima are the points labelled by 0; they project to  $\mathbb{1} \in \mathcal{X}_0$ . The maxima are the points labelled by 1 and 2, they project to the other two central elements  $e^{i\frac{2}{3}\pi}\mathbb{1}$  and  $e^{i\frac{4}{3}\pi}\mathbb{1} \in \mathcal{X}_0$ . The saddle points are situated in the middle between points labelled by 1 and 2.

The Hamiltonian equations associated with  $H$  are

$$\begin{aligned} \dot{p}_1 &= -\frac{1}{g^2\delta} \sqrt{\frac{2}{3}} \left( \sin\left(\frac{1}{\sqrt{6}}x^1\right) \cos\left(\frac{1}{\sqrt{2}}x^2\right) + \sin\left(\sqrt{\frac{2}{3}}x^1\right) \right), \\ \dot{p}_2 &= -\frac{1}{g^2\delta} \sqrt{2} \cos\left(\frac{1}{\sqrt{6}}x^1\right) \sin\left(\frac{1}{\sqrt{2}}x^2\right), \\ \dot{x}^j &= \delta^3 p_j, \quad j = 1, 2. \end{aligned} \quad (55)$$

Combining them, we obtain

$$\begin{aligned} \ddot{x}^1 + \frac{\delta^2}{g^2} \sqrt{\frac{2}{3}} \left( \sin\left(\frac{1}{\sqrt{6}}x^1\right) \cos\left(\frac{1}{\sqrt{2}}x^2\right) + \sin\left(\sqrt{\frac{2}{3}}x^1\right) \right) &= 0, \\ \ddot{x}^2 + \frac{\delta^2}{g^2} \sqrt{2} \cos\left(\frac{1}{\sqrt{6}}x^1\right) \sin\left(\frac{1}{\sqrt{2}}x^2\right) &= 0. \end{aligned} \quad (56)$$

As mentioned above, this system of equations will be studied in detail in a subsequent paper.

Next, we comment on the discussion of the dynamics in terms of the invariants of Section 5. For a given Hamiltonian function  $\tilde{H} \in C^\infty(\mathbb{R}^8)$ , dynamics takes place on  $\mathbb{R}^8$  and is ruled by the Poisson structure defined by the brackets of the coordinates (52). I.e., the equations of motion are given by

$$\dot{x}_j = \{\tilde{H}, x_j\}, \quad (x_1, \dots, x_8) = (c_0, \dots, t_3). \quad (57)$$

By construction of the Poisson structure,  $\tilde{\mathcal{P}}$  is invariant under the flow of  $\tilde{H}$  for any  $\tilde{H} \in C^\infty(\mathbb{R}^8)$ . In terms of the invariants, the Hamiltonian (2) reads

$$\tilde{H} = \frac{\delta^3}{2} t_2 + \frac{1}{g^2\delta} (3 - c_0).$$

The second term corresponds to the potential term in (54). Its level lines in  $\tilde{\mathcal{X}}$  are just straight lines parallel to the  $d_0$ -axis, cf. Figure 4. The minimum is at the corner  $(c_0, d_0) = (3, 0)$ , the maxima are at the corners  $(c_0, d_0) = (-\frac{3}{2}, \pm\sqrt{3}\frac{3}{2})$ , the saddle point is at the boundary point  $(c_0, d_0) = (-1, 0)$ .

The corresponding equations of motion (57) yield a highly coupled system, which will not be reproduced here. At first sight it does not seem to be easier to handle than the equations of motion in terms of the symplectic covering (56). It will be a future task to study and unravel this system.

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